

UNIVERSAL SUBSPACES FOR COMPACT LIE GROUPS

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ABSTRACT. For a representation of a connected compact Lie group G in a finite dimensional real vector space \mathcal{U} and a subspace \mathcal{V} of \mathcal{U} , invariant under a maximal torus of G , we obtain a sufficient condition for \mathcal{V} to meet all G -orbits in \mathcal{U} , which is also necessary in certain cases. The proof makes use of the cohomology of flag manifolds and the invariant theory of Weyl groups. Then we apply our condition to the conjugation representations of $U(n)$, $Sp(n)$, and $SO(n)$ in the space of $n \times n$ matrices over \mathbb{C} , \mathbb{H} , and \mathbb{R} , respectively. In particular, we obtain an interesting generalization of Schur's triangularization theorem.

1. INTRODUCTION

Let ρ be a representation of a connected compact Lie group G in a finite dimensional real vector space \mathcal{U} . We say that a linear subspace $\mathcal{V} \subset \mathcal{U}$ is universal if every G -orbit meets \mathcal{V} . Examples of universal subspaces occur in many places. For instance, when ρ is irreducible and nontrivial, it is easy to show (see [12]) that every hyperplane of \mathcal{U} is universal. For the adjoint representation of G in its Lie algebra \mathfrak{g} , it is well known that every Cartan subalgebra is universal. If we consider the complexified adjoint representation of G in the complexification $\mathfrak{g}_{\mathbb{C}}$ of \mathfrak{g} , then every Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is universal, and if moreover G is semisimple, then the sum of all root spaces in $\mathfrak{g}_{\mathbb{C}}$ is also universal ([8]). Questions of similar nature have been recently studied in the context of representation theory of algebraic groups, see e.g. [12, 20].

In this paper we consider the case where the stabilizer of \mathcal{V} in G contains a maximal torus T of G . Our main result (Theorem 4.2) gives a sufficient condition for the universality of \mathcal{V} , which is easy to check in concrete examples. With an additional restriction on \mathcal{V} this condition becomes also necessary (see Theorem 4.4).

Our method goes as follows. Consider the trivial vector bundle $E_{\mathcal{U}} = G/T \times \mathcal{U}$ over the flag manifold G/T . Since \mathcal{V} is T -invariant, there is a well-defined subbundle $E_{\mathcal{V}}$ of $E_{\mathcal{U}}$ whose fiber at gT is $\rho(g)(\mathcal{V})$. Every vector $u \in \mathcal{U}$, viewed as a constant section of $E_{\mathcal{U}}$, induces a section s_u of the quotient bundle $E_{\mathcal{U}}/E_{\mathcal{V}}$. It is easy to see that the G -orbit of u meets \mathcal{V} if and only if s_u has a zero. So \mathcal{V} is universal if and only if s_u has a zero for every $u \in \mathcal{U}$. A sufficient condition for this is that $E_{\mathcal{U}}/E_{\mathcal{V}}$ is orientable and has a nonzero Euler class. If we choose a T -invariant subspace \mathcal{W} of \mathcal{U} complementary to \mathcal{V} and construct similarly the subbundle $E_{\mathcal{W}}$ of $E_{\mathcal{U}}$, then $E_{\mathcal{W}} \cong E_{\mathcal{U}}/E_{\mathcal{V}}$. It is easy to see that $E_{\mathcal{W}}$ is always orientable and is equivalent to

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$G \times_T \mathcal{W}$, the associated bundle induced by the T -action on \mathcal{W} . By using a theorem of Borel on the cohomology of G/T and the invariant theory of Weyl groups, we can compute explicitly the Euler class $e(E_{\mathcal{W}})$ of $E_{\mathcal{W}}$.

Let $\mathcal{X}(T)$ be the character group of T , and let \mathcal{S} be the graded algebra of real polynomial functions on the Lie algebra of T . The map from $\mathcal{X}(T)$ to $H^2(G/T, \mathbb{R})$ sending $\chi \in \mathcal{X}(T)$ to the first Chern class of the complex line bundle $G \times_{\chi} \mathbb{C}$ induces a homomorphism $\varphi : \mathcal{S} \rightarrow H^*(G/T, \mathbb{R})$ of graded algebras, provided we double the degree in \mathcal{S} . Borel's theorem asserts that φ is surjective and its kernel is the ideal \mathcal{I} of \mathcal{S} generated by the basic generators of W -invariant polynomials, where $W = N_G(T)/T$ is the Weyl group. If we assume that the above subspace \mathcal{W} of \mathcal{U} does not contain nonzero T -invariant vectors, then \mathcal{W} can be decomposed as a direct sum $\bigoplus_{i=1}^d \mathcal{W}_i$ of 2-dimensional T -irreducible subspaces. The T -action on every \mathcal{W}_i is equivalent to the real T -module associated to some $\chi_i \in \mathcal{X}(T)$, and then $G \times_T \mathcal{W}_i$ is equivalent to the real vector bundle associated to $G \times_{\chi_i} \mathbb{C}$. So $e(E_{\mathcal{W}})$ is equal to the product of the first Chern classes of $G \times_{\chi_i} \mathbb{C}$, i.e., $\prod_{i=1}^d \varphi(\chi_i)$. We refer to the homogeneous polynomial $f_{\mathcal{V}} = \prod_{i=1}^d \chi_i$ as the characteristic polynomial of \mathcal{V} . So $e(E_{\mathcal{W}}) \neq 0$ if and only if $f_{\mathcal{V}} \notin \mathcal{I}$, and we obtain the desired sufficient condition for the universality of \mathcal{V} . In general, the question of whether $f_{\mathcal{V}}$ is in \mathcal{I} can be resolved using Gröbner bases.

An easy dimension argument shows that if \mathcal{V} is universal, then $\dim \mathcal{V} \geq \dim \mathcal{U} - \dim G/T$. The case of equality is of particular interest (indeed, for many representations, a universal subspace \mathcal{V} whose universality can be detected using our method can be shrunk to a smaller universal subspace for which the equality holds). In this case, the rank of $E_{\mathcal{W}}$ is equal to $\dim G/T$, and $f_{\mathcal{V}}$ has degree $m = \frac{1}{2} \dim G/T$. By invariant theory of Weyl groups, a homogeneous polynomial of degree m is in \mathcal{I} if and only if it is orthogonal to the fundamental harmonic polynomial f_0 (see (2.5)) with respect to a W -invariant inner product on \mathcal{S} . So if we define the characteristic number $C_{\mathcal{V}}$ of \mathcal{V} as the inner product of $f_{\mathcal{V}}$ and f_0 (up to a constant factor, which will be chosen in a way suitable for the computation of intersection numbers), then $f_{\mathcal{V}} \notin \mathcal{I}$ if and only if $C_{\mathcal{V}} \neq 0$, which gives a simpler equivalent sufficient condition for the universality of \mathcal{V} . Moreover, if we assume that the intersection indexes of the above section s_u and the zero section s_0 of $E_{\mathcal{W}}$ are all equal whenever s_u is transversal to s_0 , then \mathcal{V} is universal if and only if $C_{\mathcal{V}} \neq 0$.

In concrete examples that we shall consider, the characteristic polynomials and numbers are easy to compute. We shall do this for the conjugation representations of $U(n)$, $Sp(n)$, and $SO(n)$ in the spaces $\mathbb{M}(n, \mathbb{F})$ of $n \times n$ matrices over $\mathbb{F} = \mathbb{C}, \mathbb{H}$, and \mathbb{R} , respectively, for subspaces of $\mathbb{M}(n, \mathbb{F})$ determined by zero patterns. A zero pattern is a subset I of $\{(i, j) | i, j \in \{1, \dots, n\}, i \neq j\}$ of cardinality $n(n-1)/2$. For $\mathbb{F} = \mathbb{C}$ or \mathbb{H} and a zero pattern I , we denote by $\mathbb{M}_I(n, \mathbb{F})$ the subspace of $\mathbb{M}(n, \mathbb{F})$ consisting of matrices whose (i, j) -th entry is 0 whenever $(i, j) \in I$. We shall apply the above result to $\mathbb{M}_I(n, \mathbb{F})$ and obtain a sufficient condition for the universality of $\mathbb{M}_I(n, \mathbb{F})$ expressed as a simple property of I . In particular, we show that if I contains exactly one of (i, j) and (j, i) for $i \neq j$ (we call such I a simple zero pattern), then $\mathbb{M}_I(n, \mathbb{F})$ is universal. Similar results are also obtained for $\mathbb{F} = \mathbb{R}$. Note that if $I_0 = \{(i, j) | 1 \leq i < j \leq n\}$, $\mathbb{M}_{I_0}(n, \mathbb{C})$ is the subspace of all complex lower triangular matrices, its universality is known as Schur's triangularization theorem [25]. So our result gives a genuine generalization of Schur's theorem. If $\mathbb{F} = \mathbb{C}$ and I is a bitriangular zero pattern (defined in Section 5), the above sufficient

condition for the universality of $\mathbb{M}_I(n, \mathbb{C})$ is also necessary, and the number of flags in $U(n)/U(1)^n$ sending a generic matrix into $\mathbb{M}_I(n, \mathbb{C})$ can be computed directly from I .

The problem of universality of subspaces of $\mathbb{M}(n, \mathbb{C})$ determined by zero patterns has been raised by researchers in linear algebra. For instance, the universality of $\mathbb{M}_I(4, \mathbb{C})$ when $I = \{(1, 3), (1, 4), (2, 4), (3, 1), (4, 1), (4, 2)\}$ has been studied by several authors. It was shown by Pati [23] that it is universal. A simpler proof was given later in [5] where it is also shown that the number of flags sending a generic matrix $A \in \mathbb{M}(4, \mathbb{C})$ into $\mathbb{M}_I(n, \mathbb{C})$ is equal to 12. This zero pattern is bitriangular and so both results follow immediately from Theorem 5.2. Another simple example is the problem of universality of $\mathbb{M}_I(3, \mathbb{C})$ where $I = \{(1, 3), (2, 1), (3, 2)\}$. This was raised a few years ago [7] but remained unsettled until now. In our terminology I is a simple zero pattern, so $\mathbb{M}_I(3, \mathbb{C})$ is universal (see Theorem 5.6 and the paragraph preceding it). For further information along this direction see e.g. [6, 15].

The arrangement of this paper is as follows. In Section 2 we shall recall some preliminaries on Euler classes, intersection indexes, cohomology of flag manifolds, and invariant theory of Weyl groups. The definitions of the characteristic polynomial and the characteristic number of a subspace and their properties will be given in Section 3. The main results will be proved in Section 4. In Section 5 we shall study the universality of subspaces determined by zero patterns, generalizing Schur's triangularization theorem. Then in Section 6 we shall briefly discuss the possibility of generalizing our method and propose a question.

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2. PRELIMINARIES

In this section we recall some basic facts about Euler classes of oriented vector bundles, cohomology of the flag manifold G/T , and invariant theory of Weyl groups. Throughout this paper, we denote by G a connected compact Lie group and by T a maximal torus of G .

2.1. Euler classes and intersection indexes. Let M be a connected compact smooth n -dimensional manifold, and let E be a smooth real oriented vector bundle over M of rank r . It is well known that if the Euler class $e(E) \in H^r(M, \mathbb{Z})$ of E is nonzero, then every smooth section of E has a zero. If $E = E_1 \oplus E_2$ as oriented vector bundles, then $e(E) = e(E_1)e(E_2)$. If E admits a complex structure compatible with the orientation, then the top Chern class $c_{\text{top}}(E)$ of E is equal to $e(E)$ (see [3, 22]).

Now we assume that M is oriented and $r = n$. Let s_0 be the zero section of E . Then for any smooth section s of E , we have $e(E)([M]) = I(s, s_0)$, where $[M] \in H_n(M, \mathbb{Z})$ is the fundamental class of M and $I(s, s_0)$ is the intersection number of s and s_0 . Let $Z(s) = \{x \in M | s(x) = 0\}$ be the zero locus of s . We write $s \pitchfork_x s_0$ if s is transversal to s_0 at the point $x \in Z(s)$, and $s \pitchfork s_0$ if $s \pitchfork_x s_0$ for all $x \in Z(s)$. If $s \pitchfork s_0$, then $Z(s)$ is finite and $I(s, s_0) = \sum_{x \in Z(s)} \text{ind}_x(s)$, where $\text{ind}_x(s) = \pm 1$ is the intersection index of s and s_0 at x . Assume that E is a subbundle of a trivial bundle $M \times \mathcal{U}$, where \mathcal{U} is a real vector space, and let us view s as a map $M \rightarrow \mathcal{U}$. For $x \in Z(s)$, we can compute $\text{ind}_x(s)$ as follows. Since $s \pitchfork_x s_0$, the tangent map $(ds)_x : T_x M \rightarrow T_0 \mathcal{U} \cong \mathcal{U}$ is injective and has image E_x , the fiber of E at x . If we view $(ds)_x$ as an invertible map $T_x M \rightarrow E_x$, then $\text{ind}_x(s) = \text{sgn}((ds)_x)$. (Here the sign $\text{sgn}(A)$ of an invertible linear map $A : V_1 \rightarrow V_2$

between real oriented vector spaces refers to the sign of the determinant of the matrix of A with respect to any ordered bases of V_1 and V_2 compatible with the orientations.)

We remark that if E is the tangent bundle TM of M , then $e(TM)([M])$ is the Euler characteristic of M .

In this paper, M will almost exclusively be the flag manifold G/T . We shall need some facts about the cohomology of G/T , which we now recall.

2.2. Cohomology of G/T . Consider the maximal torus T of the connected compact Lie group G . Let $\mathcal{X}(T)$ be the character group of T . Every character $\chi \in \mathcal{X}(T)$, viewed as a representation of T in \mathbb{C} , induces a complex line bundle $L_\chi = G \times_\chi \mathbb{C}$ over G/T , where the total space of L_χ is the space of T -orbits in $G \times \mathbb{C}$ with the T -action $t \cdot (g, z) = (gt^{-1}, \chi(t)z)$. Let $c_1(L_\chi) \in H^2(G/T, \mathbb{Z})$ be the first Chern class of L_χ . Then the map sending χ to $c_1(L_\chi)$ induces a homomorphism of abelian groups

$$\mathcal{X}(T) \rightarrow H^2(G/T, \mathbb{Z}). \quad (2.1)$$

Note that $\mathcal{X}(T)$ can be identified with a lattice in \mathfrak{t}^* , the dual of the Lie algebra \mathfrak{t} of T , by using differentials. It will be more convenient to work with cohomology with real coefficients. (As $H^*(G/T, \mathbb{Z})$ is torsion free ([21], Chapter 6, Theorem 4.21), no information is lost when \mathbb{Z} is replaced by \mathbb{R} .) So we consider the linear map

$$\mathfrak{t}^* \cong \mathcal{X}(T) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow H^2(G/T, \mathbb{R}) \cong H^2(G/T, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \quad (2.2)$$

induced by the homomorphism (2.1).

Let \mathcal{S} be the symmetric algebra of \mathfrak{t}^* , identified with the graded algebra of real polynomial functions on \mathfrak{t} . Let $H^*(G/T, \mathbb{R})$ be the graded algebra of real cohomology of G/T . The homomorphism (2.2) induces an algebra homomorphism

$$\varphi : \mathcal{S} \rightarrow H^*(G/T, \mathbb{R}) \quad (2.3)$$

which doubles the degree. Let $W = N_G(T)/T$ be the Weyl group. Then W acts naturally on $\mathfrak{t}, \mathfrak{t}^*$ and \mathcal{S} . Let \mathcal{J} be the graded algebra of W -invariants in \mathcal{S} , let \mathcal{S}_d be the homogeneous component of degree d in \mathcal{S} , and let $\mathcal{J}_d = \mathcal{J} \cap \mathcal{S}_d$ and $\mathcal{J}_+ = \bigoplus_{d \geq 1} \mathcal{J}_d$. Let $\mathcal{I} = \mathcal{S}\mathcal{J}_+$ be the ideal of \mathcal{S} generated by \mathcal{J}_+ and set $\mathcal{I}_d = \mathcal{I} \cap \mathcal{S}_d$.

Theorem 2.1 (Borel [2], Theorem 26.1). *The above homomorphism φ is surjective and $\ker(\varphi) = \mathcal{I}$. Consequently, $H^*(G/T, \mathbb{R}) \cong \mathcal{S}/\mathcal{I}$ as graded algebras, provided we double the degrees in \mathcal{S}/\mathcal{I} .*

Although Borel did not mention Chern classes explicitly in his theorem in [2], his presentation is equivalent to ours. Chern classes are explicitly used in the approaches to Borel's Theorem in [1, 11, 19, 24, 27]. Our construction of the homomorphism φ follows from [19], Chapter VI, Theorem 2.1 and [27] (see also [11], Section 10.2, Proposition 3). This homomorphism can be constructed also by using classifying spaces. Let BK denote the classifying space of a group K . Then φ is equal to the composition of the Chern-Weil isomorphism $\mathcal{S} \rightarrow H^*(BT, \mathbb{R})$ and the homomorphism $H^*(BT, \mathbb{R}) \rightarrow H^*(G/T, \mathbb{R})$ induced by the fiber bundle $G/T \rightarrow BT \rightarrow BG$ (see [9, 21]).

At last, we recall the well known fact that the Euler characteristic $\chi(G/T)$ of G/T is equal to $|W|$ (see e.g. [21], Chapter 5, Theorem 3.14).

2.3. Invariant theory of Weyl groups. We have seen above that $H^*(G/T, \mathbb{R})$ can be described by W -invariants in the symmetric algebra \mathcal{S} . To understand $H^*(G/T, \mathbb{R})$ more closely, we now recall some basic facts about invariant theory of the Weyl group W . For details see [14, 17, 26].

For $H \in \mathfrak{t}$, the partial derivative $\partial_H f$ of $f \in \mathcal{S}$ is defined by $(\partial_H f)(H') = \frac{d}{dt} \big|_{t=0} f(H' + tH)$. Let (\cdot, \cdot) be a W -invariant inner product on \mathfrak{t} . Then there is a W -equivariant isomorphism $\iota : \mathfrak{t}^* \rightarrow \mathfrak{t}$ defined by the identity $(\iota(\alpha), H) = \alpha(H)$, under which the map $H \mapsto \partial_H$ extends to an isomorphism $f \mapsto \partial_f$ of \mathcal{S} onto the algebra of differential operators on \mathfrak{t} with constant coefficients. The bilinear form B on \mathcal{S} defined by $B(f, g) = (\partial_f g)(0)$ is a W -invariant inner product on \mathcal{S} . If $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathfrak{t}^*$, then

$$B(\alpha_1 \cdots \alpha_k, \beta_1 \cdots \beta_k) = \sum_{\sigma \in S_k} \prod_{i=1}^k (\iota(\alpha_i), \iota(\beta_{\sigma(i)})), \quad (2.4)$$

where S_k is the symmetric group on k letters. It is easily seen that ∂_f is the adjoint operator of the multiplication operator by f , that is, $B(g, fh) = B(\partial_f g, h)$ for any $f, g, h \in \mathcal{S}$.

A polynomial $f \in \mathcal{S}$ is W -harmonic if $\partial_g f = 0$ for all $g \in \mathcal{J}_+$. Let \mathcal{H} denote the space of W -harmonic polynomials, and let $\mathcal{H}_d = \mathcal{H} \cap \mathcal{S}_d$. Let $\Phi \subset \mathcal{X}(T) \subset \mathfrak{t}^*$ be the root system of G , Φ^+ a system of positive roots, and $m = |\Phi^+| = \frac{1}{2} \dim G/T$. Some basic facts that we are going to use are collected in the following theorem.

Theorem 2.2. (1) *There are $r = \dim \mathfrak{t}$ basic homogeneous W -invariant polynomials F_1, \dots, F_r which are algebraically independent such that the subalgebra of \mathcal{S} generated by F_1, \dots, F_r is \mathcal{J} , and the ideal of \mathcal{S} generated by F_1, \dots, F_r is $\mathcal{I} = \mathcal{S}\mathcal{J}_+$.*
(2) *The degrees $d_i = \deg F_i$ are determined by W , $\prod_{i=1}^r d_i = |W|$, $\sum_{i=1}^r (d_i - 1) = m$.*
(3) *$\mathcal{S} = \mathcal{I} \oplus \mathcal{H}$ orthogonally with respect to B , and the Poincaré polynomial of \mathcal{S}/\mathcal{I} is given by $\sum_{d=1}^{\infty} (\dim \mathcal{H}_d) t^d = \prod_{i=1}^r (1 + t + \cdots + t^{d_i-1})$. In particular, $\mathcal{H}_d = 0$ for $d > m$ and $\dim \mathcal{H}_m = 1$.*
(4) *The polynomial*

$$f_0 = \prod_{\alpha \in \Phi^+} \alpha \quad (2.5)$$

is in \mathcal{H}_m , and $\mathcal{H} = \{\partial_f f_0 \mid f \in \mathcal{S}\}$.

(5) *For $f \in \mathcal{S}$, $f \in \mathcal{I}$ if and only if $\partial_f f_0 = 0$. In particular, for $f \in \mathcal{S}_m$ we have $f \in \mathcal{I}_m$ if and only if $B(f, f_0) = 0$.*

(6) *$f \in \mathcal{S}$ is W -skew if and only if $f \in f_0 \mathcal{J}$. (Here f is called W -skew if $w \cdot f = \text{sgn}(w)f$ for every $w \in W$.)*

We refer to the polynomial $f_0 \in \mathcal{H}_m$ given in (2.5) as the *fundamental harmonic polynomial*. Given $f \in \mathcal{S}_m$, it is important to know when $f \in \mathcal{I}_m$. Theorem 2.2 (5) provides a characterization using f_0 and the inner product B . Although it is not difficult to compute B using (2.4), the following generalization of Theorem 2.2 (5) will be more convenient.

Proposition 2.3. $\mathcal{I}_m = f_0^\perp$ with respect to any W -invariant inner product on \mathcal{S}_m .

Proof. By Theorem 2.2 (6), every W -skew polynomial in \mathcal{S}_m is a multiple of f_0 . So for the natural representation of W in \mathcal{S}_m , the isotypic component of the irreducible

representation $w \mapsto \text{sgn}(w)$ is $\mathbb{R}f_0$. Hence the orthogonal complement of $\mathbb{R}f_0$ is independent of the choice of the W -invariant inner product on \mathcal{S}_m . \square

In view of Proposition 2.3, the fundamental harmonic polynomial f_0 plays an important role. For later reference, we give below explicit expressions for f_0 and a W -invariant inner product on \mathcal{S} for classical groups. Before doing this, we define a polynomial λ_0 in $\mathbb{R}[x_1, \dots, x_n]$ as

$$\lambda_0(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n x_{\sigma(i)}^{n-i}, \quad (2.6)$$

where $x = (x_1, \dots, x_n)$. This polynomial will appear in all cases of classical groups.

Example 2.1. Let $G = U(n)$, let $T = U(1)^n = \{t = \text{diag}(t_1, \dots, t_n) | t_i \in U(1)\}$ be the standard maximal torus and $\mathfrak{t} = \{x = \text{diag}(\sqrt{-1}x_1, \dots, \sqrt{-1}x_n) | x_i \in \mathbb{R}\}$. Then \mathcal{S} can be identified with the polynomial algebra $\mathbb{R}[x_1, \dots, x_n]$, and a character $\chi(t) = t_1^{m_1} \cdots t_n^{m_n}$ with the linear form $m_1x_1 + \cdots + m_nx_n$. The root system is given by

$$\Phi = \{\pm(x_i - x_j) | 1 \leq i < j \leq n\},$$

and Φ^+ can be chosen as

$$\Phi^+ = \{x_i - x_j | 1 \leq i < j \leq n\}.$$

So in this case

$$f_0(x) = \lambda_0(x).$$

We have $\deg f_0 = |\Phi^+| = n(n-1)/2$ and $W \cong S_n$. There is a unique W -invariant inner product on \mathcal{S} for which the monomials $x_1^{m_1} \cdots x_n^{m_n}$ form an orthonormal basis. The basic generators F_1, \dots, F_n of \mathcal{I} can be chosen as the elementary symmetric polynomials in x_1, \dots, x_n . \square

Example 2.2. Let $G = Sp(n)$. $T = U(1)^n$ is a maximal torus and $\mathfrak{t} = \{x = \text{diag}(\sqrt{-1}x_1, \dots, \sqrt{-1}x_n) | x_i \in \mathbb{R}\}$. So $\mathcal{S} \cong \mathbb{R}[x_1, \dots, x_n]$. The root system is given by

$$\Phi = \{\pm x_i \pm x_j | 1 \leq i < j \leq n\} \cup \{\pm 2x_i | 1 \leq i \leq n\},$$

and Φ^+ can be chosen as

$$\Phi^+ = \{x_i \pm x_j | 1 \leq i < j \leq n\} \cup \{2x_i | 1 \leq i \leq n\}.$$

So

$$f_0(x) = 2^n \lambda_0(x_1^2, \dots, x_n^2) \prod_{i=1}^n x_i.$$

We have $\deg f_0 = |\Phi^+| = n^2$ and $W \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. The W -invariant inner product on \mathcal{S} can be chosen as in Example 2.1. The basic generators F_1, \dots, F_n of \mathcal{I} can be chosen as the elementary symmetric polynomials in x_1^2, \dots, x_n^2 . \square

Example 2.3. Let $G = SO(2n)$. $T = SO(2)^n$ is a maximal torus and $\mathfrak{t} = \left\{x = \text{diag} \left(\begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & x_n \\ -x_n & 0 \end{pmatrix} \right) \right\}$. So $\mathcal{S} \cong \mathbb{R}[x_1, \dots, x_n]$. The root system is given by

$$\Phi = \{\pm x_i \pm x_j | 1 \leq i < j \leq n\},$$

and Φ^+ can be chosen as

$$\Phi^+ = \{x_i \pm x_j | 1 \leq i < j \leq n\}.$$

So

$$f_0(x) = \lambda_0(x_1^2, \dots, x_n^2).$$

We have $\deg f_0 = |\Phi^+| = n(n-1)$ and $W \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$. The W -invariant inner product on \mathcal{S} can be chosen as in Example 2.1. The basic generators F_1, \dots, F_n of \mathcal{I} can be chosen as the first $n-1$ elementary symmetric polynomials in x_1^2, \dots, x_n^2 and $\prod_{i=1}^n x_i$. \square

Example 2.4. Let $G = SO(2n+1)$. $T = SO(2)^n \times \{1\}$ is a maximal torus and $\mathfrak{t} = \left\{ x = \text{diag} \left(\begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & x_n \\ -x_n & 0 \end{pmatrix}, 0 \right) \right\}$. So $\mathcal{S} \cong \mathbb{R}[x_1, \dots, x_n]$. The root system is given by

$$\Phi = \{\pm x_i \pm x_j | 1 \leq i < j \leq n\} \cup \{\pm x_i | 1 \leq i \leq n\},$$

and Φ^+ can be chosen as

$$\Phi^+ = \{x_i \pm x_j | 1 \leq i < j \leq n\} \cup \{x_i | 1 \leq i \leq n\}.$$

So

$$f_0(x) = \lambda_0(x_1^2, \dots, x_n^2) \prod_{i=1}^n x_i.$$

We have $\deg f_0 = |\Phi^+| = n^2$ and $W \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. The W -invariant inner product on \mathcal{S} can be chosen as in Example 2.1. The basic generators F_1, \dots, F_n of \mathcal{I} can be chosen to be the same as in Example 2.2. \square

3. CHARACTERISTIC POLYNOMIALS OF SUBSPACES

Let ρ be a representation of G in a finite dimensional real vector space \mathcal{U} , and let \mathcal{U}^T be the subspace of T -invariant vectors in \mathcal{U} . Let $\mathcal{V} \subset \mathcal{U}$ be a T -invariant subspace containing \mathcal{U}^T . We shall now define the characteristic polynomial $f_{\mathcal{V}}$ of \mathcal{V} (up to a sign which depends on the orientation of \mathcal{U}/\mathcal{V}).

Let \mathcal{W} be a T -invariant subspace of \mathcal{U} complementary to \mathcal{V} . Every real irreducible representation of T is either trivial or 2-dimensional. In the latter case, it is equivalent to the real T -module associated to some nontrivial character χ of T . Since $\mathcal{U}^T \subset \mathcal{V}$, there is no trivial subrepresentation of T in \mathcal{W} . So \mathcal{W} can be decomposed as a direct sum

$$\mathcal{W} = \bigoplus_{i=1}^d \mathcal{W}_i, \quad d = \frac{1}{2} \dim \mathcal{W}, \quad (3.1)$$

of 2-dimensional T -irreducible subspaces. Note that the real T -modules associated to χ and $-\chi$ are equivalent. To associate to \mathcal{W}_i a unique $\chi_i \in \mathcal{X}(T) \setminus \{0\}$, we choose and fix an orientation on \mathcal{W}_i . Then there is a unique $\chi_i \in \mathcal{X}(T) \setminus \{0\}$ for which there exists a T -equivariant orientation preserving isomorphism $\sigma_i : (\mathbb{R}^2, \chi_i) \rightarrow \mathcal{W}_i$, where (\mathbb{R}^2, χ_i) is the real T -module associated to χ_i with the standard orientation. The orientations of the \mathcal{W}_i 's induce an orientation on $\mathcal{W} \cong \mathcal{U}/\mathcal{V}$. It is easy to see that the product of the χ_i 's depends only on the induced orientation on \mathcal{W} , and changes sign if the orientation is reversed.

Definition 3.1. Given the orientation on $\mathcal{W} \cong \mathcal{U}/\mathcal{V}$ as above, we define the *characteristic polynomial* $f_{\mathcal{V}} \in \mathcal{S}_d$ of \mathcal{V} by

$$f_{\mathcal{V}} = \prod_{i=1}^d \chi_i.$$

The equivalence σ_i above induces a complex structure J_i on \mathcal{W}_i , which is independent of the choice of σ_i and is determined by the orientation and the T -action on \mathcal{W}_i . The direct sum

$$J_{\mathcal{W}} = \bigoplus_{i=1}^d J_i \quad (3.2)$$

is a complex structure on \mathcal{W} compatible with the orientation. For any nonzero vectors $w_i \in \mathcal{W}_i$, the ordered basis

$$\{w_1, J_{\mathcal{W}}w_1, \dots, w_d, J_{\mathcal{W}}w_d\} \quad (3.3)$$

of \mathcal{W} is compatible with the orientation.

Remark 3.1. If \mathcal{U} admits a complex structure such that the G -action is complex linear, and if $\mathcal{V} \subset \mathcal{U}$ is a complex subspace, then we can choose \mathcal{W} to be a complex subspace. Thus \mathcal{W} has the complex decomposition

$$\mathcal{W} = \bigoplus_{\chi \in \mathcal{X}(T)} \mathcal{W}_{\chi},$$

where \mathcal{W}_{χ} is the χ -isotypic component of \mathcal{W} . As $\mathcal{U}^T \subset \mathcal{V}$, we have $\mathcal{W}_0 = 0$. Each \mathcal{W}_{χ} can be further decomposed as a direct sum of complex lines, which serve as the \mathcal{W}_i 's in (3.1). If the orientation on each of these complex lines is chosen to be the one induced by the original complex structure, then

$$f_{\mathcal{V}} = \prod_{\chi \in \mathcal{X}(T)} \chi^{\dim_{\mathbb{C}} \mathcal{W}_{\chi}},$$

and $J_{\mathcal{W}}$ coincides with the original complex structure on \mathcal{W} .

We now consider the trivial real vector bundle $E_{\mathcal{U}} = G/T \times \mathcal{U}$ over the flag manifold G/T . For any T -invariant subspace \mathcal{L} of \mathcal{U} , there is a well-defined subbundle $E_{\mathcal{L}}$ of $E_{\mathcal{U}}$ whose fiber at gT is $\rho(g)(\mathcal{L})$. In particular, \mathcal{V} and \mathcal{W} are T -invariant, and so we have the subbundles $E_{\mathcal{V}}$ and $E_{\mathcal{W}}$. Since $\mathcal{U} = \rho(g)(\mathcal{V}) \oplus \rho(g)(\mathcal{W})$ for every $g \in G$, we have $E_{\mathcal{U}} = E_{\mathcal{V}} \oplus E_{\mathcal{W}}$. The T -action on \mathcal{W} preserves the orientation. So the orientation on \mathcal{W} induces an orientation on $E_{\mathcal{W}} \cong E_{\mathcal{U}}/E_{\mathcal{V}}$, and we can consider the Euler class $e(E_{\mathcal{W}}) = e(E_{\mathcal{U}}/E_{\mathcal{V}}) \in H^{2d}(G/T, \mathbb{R})$. Since $J_{\mathcal{W}}$ is T -invariant, it induces a complex structure on $E_{\mathcal{W}}$. So we can also consider the top Chern class $c_{\text{top}}(E_{\mathcal{W}}) \in H^{2d}(G/T, \mathbb{R})$. Since the orientation and the complex structure are compatible, $e(E_{\mathcal{W}}) = c_{\text{top}}(E_{\mathcal{W}})$.

Theorem 3.1. *We have $e(E_{\mathcal{U}}/E_{\mathcal{V}}) = \varphi(f_{\mathcal{V}})$, where φ is the homomorphism (2.3).*

Proof. We view $E_{\mathcal{W}}$ as a complex vector bundle and prove that $c_{\text{top}}(E_{\mathcal{W}}) = \varphi(f_{\mathcal{V}})$. Let L_i be the subbundle of $E_{\mathcal{U}}$ whose fiber at gT is $\rho(g)(\mathcal{W}_i)$. The complex structure J_i on \mathcal{W}_i induces a complex structure on L_i . By (3.1), $E_{\mathcal{W}} \cong \bigoplus_{i=1}^d L_i$ as complex vector bundles. So we have

$$c_{\text{top}}(E_{\mathcal{W}}) = \prod_{i=1}^d c_1(L_i), \quad (3.4)$$

where $c_1(L_i)$ is the first Chern class of L_i . For $\chi \in \mathcal{X}(T)$ let $L_\chi = G \times_\chi \mathbb{C}$ be the complex line bundle over G/T induced by χ . We claim that $L_i \cong L_{\chi_i}$ as complex line bundles. Indeed, given any nonzero vector $w_i \in \mathcal{W}_i$, the map $G \times \mathbb{C} \rightarrow L_i$ defined by

$$(g, z) \mapsto (gT, \rho(g)(zw_i))$$

induces an equivalence between L_{χ_i} and L_i . So by the definition of φ , we have

$$c_1(L_i) = c_1(L_{\chi_i}) = \varphi(\chi_i). \quad (3.5)$$

From (3.4) and (3.5) we deduce that

$$c_{\text{top}}(E_{\mathcal{W}}) = \prod_{i=1}^d \varphi(\chi_i) = \varphi\left(\prod_{i=1}^d \chi_i\right) = \varphi(f_{\mathcal{V}}).$$

□

Set $m = \frac{1}{2} \dim G/T$. We shall be mainly interested in the case $d = m$. In this case, it will be useful to know the explicit value of $e(E_{\mathcal{U}}/E_{\mathcal{V}})([G/T])$, where $[G/T] \in H_{2m}(G/T, \mathbb{Z})$ is the fundamental class of G/T with respect to some orientation on G/T . We fix the orientation on G/T as follows. Let \mathfrak{g} and \mathfrak{t} be the Lie algebras of G and T , respectively, and let $\Phi \subset \mathcal{X}(T)$ be the root system of G and $\Phi^+ = \{\alpha_1, \dots, \alpha_m\}$ a system of positive roots. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$ be the root space decomposition of the complexification of \mathfrak{g} , and set $\mathfrak{m} = \mathfrak{g} \cap (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha})$. Then there exist $X_{\alpha} \in \mathfrak{g}_{\alpha}$ such that

$$\{X_{\alpha_1} - X_{-\alpha_1}, \sqrt{-1}(X_{\alpha_1} + X_{-\alpha_1}), \dots, X_{\alpha_m} - X_{-\alpha_m}, \sqrt{-1}(X_{\alpha_m} + X_{-\alpha_m})\} \quad (3.6)$$

is a basis of \mathfrak{m} . For $\alpha \in \Phi^+$, let $\mathfrak{m}_{\alpha} = \text{span}\{X_{\alpha} - X_{-\alpha}, \sqrt{-1}(X_{\alpha} + X_{-\alpha})\}$, and let J_{α} be the complex structure on \mathfrak{m}_{α} defined by

$$J_{\alpha}(X_{\alpha} - X_{-\alpha}) = \sqrt{-1}(X_{\alpha} + X_{-\alpha}), \quad J_{\alpha}(\sqrt{-1}(X_{\alpha} + X_{-\alpha})) = -(X_{\alpha} - X_{-\alpha}).$$

Then $\mathfrak{m} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{m}_{\alpha}$, and $J_{\mathfrak{m}} = \bigoplus_{\alpha \in \Phi^+} J_{\alpha}$ is a complex structure on \mathfrak{m} , which induces an orientation on \mathfrak{m} compatible with the ordered basis (3.6). For $g \in G$, define the map $\tau_g : \mathfrak{m} \rightarrow G/T$ by

$$\tau_g(X) = ge^X T. \quad (3.7)$$

Then the differential $(d\tau_g)_0 : \mathfrak{m} \rightarrow T_{gT}(G/T)$ is invertible and induces an orientation on $T_{gT}(G/T)$, which depends only on gT . This induces an orientation on G/T .

Definition 3.2. If $d = m$, we define the *characteristic number* $C_{\mathcal{V}}$ of \mathcal{V} by

$$C_{\mathcal{V}} = \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} |W|,$$

where W is the Weyl group, $\langle \cdot, \cdot \rangle$ is a W -invariant inner product on \mathcal{S}_m , and f_0 is the fundamental harmonic polynomial defined in (2.5).

Proposition 3.2. $C_{\mathcal{V}}$ is independent of the choice of $\langle \cdot, \cdot \rangle$.

Proof. The linear functional $\mathcal{S}_m \rightarrow \mathbb{R}$ sending f to $\frac{\langle f, f_0 \rangle}{\langle f_0, f_0 \rangle} |W|$ takes the value $|W|$ at f_0 and vanishes on $f_0^\perp = \mathcal{I}_m$. □

We shall prove that, with respect to the above orientation on G/T , $e(E_{\mathcal{W}})([G/T]) = C_{\mathcal{V}}$. We first prove a special case. Let $E_{\mathfrak{m}}$ be the subbundle of the trivial bundle $G/T \times \mathfrak{g}$ corresponding to the T -invariant subspace \mathfrak{m} .

Lemma 3.3. *For the adjoint representation of G in \mathfrak{g} , we have $f_{\mathfrak{t}} = f_0$ and $e(E_{\mathfrak{m}})([G/T]) = C_{\mathfrak{t}} = |W|$.*

Proof. With respect to the complex structure J_{α} , the T -action on \mathfrak{m}_{α} is complex and is equivalent to α . So $f_{\mathfrak{t}} = \prod_{\alpha \in \Phi^+} \alpha = f_0$ and $C_{\mathfrak{t}} = |W|$.

To finish the proof, we note that the orientation preserving map

$$(d\tau_g)_0 \circ \text{Ad}(g)^{-1} : (E_{\mathfrak{m}})_{gT} = \text{Ad}(g)(\mathfrak{m}) \rightarrow T_{gT}(G/T)$$

depends only on gT , where τ_g is the map defined in (3.7). This induces an orientation preserving equivalence between $E_{\mathfrak{m}}$ and the tangent bundle $T(G/T)$. So $e(E_{\mathfrak{m}})([G/T]) = e(T(G/T))(G/T) = \chi(G/T) = |W|$. \square

Theorem 3.4. *We have $e(E_{\mathcal{U}}/E_{\mathcal{V}})([G/T]) = C_{\mathcal{V}}$.*

Proof. By Theorem 2.1 and Proposition 2.3,

$$f_{\mathcal{V}} - \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} f_0 \in f_0^{\perp} = \mathcal{I}_{\mathfrak{m}} \subset \ker(\varphi).$$

So by Theorem 3.1 and Lemma 3.3, we have

$$\begin{aligned} e(E_{\mathcal{W}})([G/T]) &= \varphi(f_{\mathcal{V}})([G/T]) = \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} \varphi(f_0)([G/T]) \\ &= \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} e(E_{\mathfrak{m}})([G/T]) = \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} |W| = C_{\mathcal{V}}. \end{aligned}$$

\square

4. UNIVERSAL SUBSPACES

Given the representation of G in \mathcal{U} , we say that a linear subspace \mathcal{V} of \mathcal{U} is *universal* if every G -orbit in \mathcal{U} meets \mathcal{V} . Let $G_{\mathcal{V}}$ be the stabilizer of \mathcal{V} in G .

Lemma 4.1. *If \mathcal{V} is universal, then $\dim \mathcal{V} \geq \dim \mathcal{U} - \dim G/G_{\mathcal{V}}$.*

Proof. Let $G \times_{G_{\mathcal{V}}} \mathcal{V}$ be the vector bundle over $G/G_{\mathcal{V}}$ whose total space is the space of $G_{\mathcal{V}}$ -orbits in $G \times \mathcal{V}$ with the $G_{\mathcal{V}}$ -action $h \cdot (g, v) = (gh^{-1}, \rho(h)(v))$. Then the map $F : G \times \mathcal{V} \rightarrow \mathcal{U}$, $F(g, v) = \rho(g)(v)$ reduces to a map $\tilde{F} : G \times_{G_{\mathcal{V}}} \mathcal{V} \rightarrow \mathcal{U}$. Since \mathcal{V} is universal, F and \tilde{F} are surjective. So $\dim G/G_{\mathcal{V}} + \dim \mathcal{V} = \dim(G \times_{G_{\mathcal{V}}} \mathcal{V}) \geq \dim \mathcal{U}$. \square

Assume that \mathcal{V} is T -invariant, that is, $T \subset G_{\mathcal{V}}$. By the above lemma, if \mathcal{V} is universal, then $\dim \mathcal{V} \geq \dim \mathcal{U} - 2m$, where $m = \frac{1}{2} \dim G/T$. We say that \mathcal{V} has *optimal dimension* if $\dim \mathcal{V} = \dim \mathcal{U} - 2m$. In this case the characteristic polynomial $f_{\mathcal{V}}$ of \mathcal{V} has degree $d = m$.

Theorem 4.2. *Let ρ be a representation of G in a finite dimensional real vector space \mathcal{U} , and let \mathcal{V} be a T -invariant subspace of \mathcal{U} such that $\mathcal{U}^T \subset \mathcal{V}$. Then $f_{\mathcal{V}} \notin \mathcal{I}$ implies that \mathcal{V} is universal. In particular, if \mathcal{V} has optimal dimension and $C_{\mathcal{V}} \neq 0$, then \mathcal{V} is universal.*

Proof. Let $E_{\mathcal{U}}, E_{\mathcal{V}}$ be the vector bundles defined in Section 3. Every vector $u \in \mathcal{U}$, viewed as a constant section of $E_{\mathcal{U}}$, induces a smooth section s_u of $E_{\mathcal{U}}/E_{\mathcal{V}}$. It is obvious that gT is a zero of s_u if and only if $u \in \rho(g)(\mathcal{V})$, that is, $\rho(g^{-1})(u) \in \mathcal{V}$. So \mathcal{V} is universal if and only if for any $u \in \mathcal{U}$, s_u has a zero. This will be ensured if the Euler class $e(E_{\mathcal{U}}/E_{\mathcal{V}})$ is nonzero.

By Theorem 3.1, $e(E_{\mathcal{U}}/E_{\mathcal{V}}) = \varphi(f_{\mathcal{V}})$, where $\varphi : \mathcal{S} \rightarrow H^*(G/T, \mathbb{R})$ is the homomorphism (2.3). By Theorem 2.1, the kernel of φ is \mathcal{I} . So $e(E_{\mathcal{U}}/E_{\mathcal{V}}) \neq 0$ if and only if $f_{\mathcal{V}} \notin \mathcal{I}$. This proves the main assertion. The particular case follows from Proposition 2.3. \square

In Theorem 4.2, when \mathcal{V} has optimal dimension, the sufficient condition $C_{\mathcal{V}} \neq 0$ for the universality of \mathcal{V} is fairly easy to check. If $\dim \mathcal{V} > \dim \mathcal{U} - 2m$, then $\deg f_{\mathcal{V}} < m$. In this case, to determine whether $f_{\mathcal{V}} \notin \mathcal{I}$, one can use Theorem 2.2 (5). Since \mathcal{I} is generated by the basic generators F_1, \dots, F_r , one can also use the theory of Gröbner bases (see [4]). But for many representations, this case can be reduced to the case of optimal dimension as the next proposition shows. (In a special case, this result was pointed out to us by J.-K. Yu.)

Proposition 4.3. *Suppose that the identity component of $\ker(\rho)$ is a torus, and suppose that there exists a G -orbit \mathcal{O} in \mathcal{U} such that $\dim \mathcal{O} = \dim G/\ker(\rho)$. If \mathcal{V} is a T -invariant subspace of \mathcal{U} such that $\mathcal{U}^T \subset \mathcal{V}$ and $f_{\mathcal{V}} \notin \mathcal{I}$, then there exists a T -invariant subspace \mathcal{V}' of \mathcal{U} with optimal dimension such that $\mathcal{U}^T \subset \mathcal{V}' \subset \mathcal{V}$ and $f_{\mathcal{V}'} \notin \mathcal{I}$.*

Proof. By replacing G with $G/\ker(\rho)$ if necessary, we may assume that ρ is faithful. If \mathcal{V} has optimal dimension, there is nothing to prove. So we assume that $d = \deg f_{\mathcal{V}} < m$. Let $Z = \{\beta \in \mathfrak{t}^* \mid f_{\mathcal{V}}\beta \in \mathcal{I}\}$. Then Z is a subspace of \mathfrak{t}^* . We claim that $Z \neq \mathfrak{t}^*$. Indeed, if $Z = \mathfrak{t}^*$, then $f_{\mathcal{V}}\mathfrak{t}^* \subset \mathcal{I}$, and then $f_{\mathcal{V}}\mathcal{S}_{m-d} \subset \mathcal{I}$. Hence $\varphi(f_{\mathcal{V}})H^{2(m-d)}(G/T, \mathbb{R}) = \varphi(f_{\mathcal{V}})\varphi(\mathcal{S}_{m-d}) = 0$. But by the Poincaré duality, the pairing $H^{2d}(G/T, \mathbb{R}) \times H^{2(m-d)}(G/T, \mathbb{R}) \rightarrow H^{2m}(G/T, \mathbb{R}) \cong \mathbb{R}$ defined by $(\xi_1, \xi_2) \mapsto \xi_1\xi_2$ is nondegenerate. As $\varphi(f_{\mathcal{V}}) \neq 0$, we have a contradiction.

We decompose \mathcal{V} as $\mathcal{V} = \mathcal{U}^T \oplus \bigoplus_{i=1}^{d'} \mathcal{V}_i$, where each \mathcal{V}_i is 2-dimensional and T -invariant, and the T -action on \mathcal{V}_i is equivalent to the real T -module associated to some $\chi'_i \in \mathcal{X}(T)$. Since $f_{\mathcal{V}} \notin \mathcal{I}$, \mathcal{V} is universal. So $\mathcal{O} \cap \mathcal{V} \neq \emptyset$ and we can choose $v \in \mathcal{O} \cap \mathcal{V}$. Then for any $H \in \bigcap_{i=1}^{d'} \ker(\chi'_i)$ and $t \in \mathbb{R}$, we have $\rho(e^{tH})(v) = v$. But $\dim \mathcal{O} = \dim G$ forces $H = 0$. So we have $\bigcap_{i=1}^{d'} \ker(\chi'_i) = 0$, that is, $\text{span}\{\chi'_1, \dots, \chi'_{d'}\} = \mathfrak{t}^*$. Since $Z \neq \mathfrak{t}^*$, there exists i_0 such that $\chi'_{i_0} \notin Z$, that is, $f_{\mathcal{V}}\chi'_{i_0} \notin \mathcal{I}$. Then, for $\mathcal{V}_1 = \mathcal{U}^T \oplus \bigoplus_{i \neq i_0} \mathcal{V}_i$, we have $f_{\mathcal{V}_1} = f_{\mathcal{V}}\chi'_{i_0} \notin \mathcal{I}$. Now the proposition follows by induction. \square

Remark 4.1. By the principal orbit theorem (see e.g. [18]), the principal orbits have the maximal dimension and their union is an open and dense subset of \mathcal{U} . Hence there exists a G -orbit \mathcal{O} with $\dim \mathcal{O} = \dim G/\ker(\rho)$ if and only if the dimension of the principal orbits is $\dim G/\ker(\rho)$. Many representations satisfy this condition. For instance, the condition is satisfied by the complexified adjoint representation of G . Indeed, there are only finitely many real irreducible representations of G for which the condition fails. Moreover, in the case when G is simple, a complete list of real irreducible representations for which the condition fails is given in [16]. A similar list for complex representations (not necessarily irreducible) of complex simple Lie groups is given in [10].

It is interesting to ask to what extent the converse of Theorem 4.2 holds. We now prove a special case. Further discussion will be given in Section 6.

Suppose that \mathcal{V} has optimal dimension, and let \mathcal{W} be a T -invariant subspace of \mathcal{U} complementary to \mathcal{V} . Let \mathfrak{g} and \mathfrak{m} be as before. For every $v \in \mathcal{V}$, we define a

map $\psi_v : \mathfrak{m} \rightarrow \mathcal{W}$ by

$$\psi_v(X) = -P_{\mathcal{W}}d\rho(X)(v), \quad (4.1)$$

where $P_{\mathcal{W}}$ is the projection of \mathcal{U} on \mathcal{W} along \mathcal{V} , and $d\rho : \mathfrak{g} \rightarrow \text{End}(\mathcal{U})$ is the differential of ρ . Note that since \mathcal{V} has optimal dimension, we have $\dim \mathfrak{m} = \dim \mathcal{W}$. Let $\det(\psi_v)$ be the determinant of the matrix of ψ_v with respect to the basis (3.6) of \mathfrak{m} and the basis

$$\{w_1, J_{\mathcal{W}}w_1, \dots, w_m, J_{\mathcal{W}}w_m\} \quad (4.2)$$

of \mathcal{W} (see (3.3)). The map $v \mapsto \det(\psi_v)$ is a real polynomial function on \mathcal{V} .

Theorem 4.4. *Let the notation be as above. Suppose that $\det(\psi_v)$ does not take both positive and negative values on \mathcal{V} . Then the following statements are equivalent:*

- (1) $C_{\mathcal{V}} \neq 0$;
- (2) \mathcal{V} is universal;
- (3) $\det(\psi_v)$ is not identically zero.

Before proving the theorem, we first make some preparations. Recall from the proof of Theorem 4.2 that we associated to each $u \in \mathcal{U}$ a section s_u of the oriented bundle $E_{\mathcal{U}}/E_{\mathcal{V}}$ over G/T , and that a point $gT \in G/T$ lies in the zero locus $Z(s_u)$ of s_u if and only if $\rho(g^{-1})(u) \in \mathcal{V}$. We denote the G -orbit of u by \mathcal{O}_u .

Lemma 4.5. *Let $u \in \mathcal{U}$, $x = gT \in Z(s_u)$ and $v = \rho(g^{-1})(u)$. Then*

- (1) $\text{ind}_x(s_u) = \text{sgn}(\psi_v)$;
- (2) $\mathcal{O}_u \pitchfork_v \mathcal{V} \Leftrightarrow s_u \pitchfork_x s_0 \Leftrightarrow \det(\psi_v) \neq 0$. In particular, $\mathcal{O}_u \pitchfork \mathcal{V}$ if and only if $s_u \pitchfork s_0$.

Proof. (1). Since $E_{\mathcal{U}}/E_{\mathcal{V}}$ is equivalent to $E_{\mathcal{W}}$, s_u can be viewed as a map $s_u : G/T \rightarrow \mathcal{U}$ and can be expressed as

$$s_u(gT) = P_{\rho(g)(\mathcal{W})}(u) = \rho(g)P_{\mathcal{W}}\rho(g^{-1})(u),$$

where $P_{\rho(g)(\mathcal{W})}$ is the projection of \mathcal{U} onto $\rho(g)(\mathcal{W})$ along $\rho(g)(\mathcal{V})$. Since $x \in Z(s_u)$, we can view the tangent map $(ds_u)_x$ as a map from $T_x(G/T)$ into $\rho(g)(\mathcal{W})$. Let τ_g be the map defined in (3.7). Then we have

$$s_u \circ \tau_g(X) = \rho(g)\rho(e^X)P_{\mathcal{W}}\rho(e^{-X})\rho(g^{-1})(u) = \rho(g)e^{d\rho(X)}P_{\mathcal{W}}e^{-d\rho(X)}(v).$$

Since $v \in \mathcal{V}$,

$$\begin{aligned} (ds_u)_x \circ (d\tau_g)_0(X) &= \rho(g)(d\rho(X)P_{\mathcal{W}}(v) - P_{\mathcal{W}}d\rho(X)(v)) \\ &= -\rho(g)P_{\mathcal{W}}d\rho(X)(v) = \rho(g)\psi_v(X), \end{aligned}$$

that is, $(ds_u)_x \circ (d\tau_g)_0 = \rho(g) \circ \psi_v$. Since both $(d\tau_g)_0$ and $\rho(g) : \mathcal{W} \rightarrow \rho(g)(\mathcal{W})$ are invertible and preserve the orientation, we have $\text{ind}_x(s_u) = \text{sgn}((ds_u)_x) = \text{sgn}(\psi_v)$.

(2). The assertion $s_u \pitchfork_x s_0 \Leftrightarrow \det(\psi_v) \neq 0$ follows directly from (1), and the assertion $\mathcal{O}_u \pitchfork_v \mathcal{V} \Leftrightarrow \det(\psi_v) \neq 0$ is obvious from the definition of ψ_v . \square

We say that a flag $gT \in G/T$ sends u into \mathcal{V} if $gT \in Z(s_u)$. By the above lemma, if $\mathcal{O}_u \pitchfork \mathcal{V}$, then the number

$$N(u, \mathcal{V}) = \#Z(s_u)$$

of flags sending u into \mathcal{V} is finite. By the transversality theorem (see [13]), the vectors $u \in \mathcal{U}$ for which $\mathcal{O}_u \pitchfork \mathcal{V}$ form an open and dense subset of \mathcal{U} . So $N(u, \mathcal{V})$ is finite for generic $u \in \mathcal{U}$. In general, $N(u, \mathcal{V})$ is not constant even for generic u . But this is the case if the condition of Theorem 4.4 holds.

Lemma 4.6. *Under the condition of Theorem 4.4, for $u \in \mathcal{U}$ with $\mathcal{O}_u \pitchfork \mathcal{V}$, we have $N(u, \mathcal{V}) = |C_{\mathcal{V}}|$.*

Proof. Without loss of generality, we may assume that $\det(\psi_v) \geq 0$ for every $v \in \mathcal{V}$. Since $\mathcal{O}_u \pitchfork \mathcal{V}$, for every $x = gT \in Z(s_u)$ and $v = \rho(g^{-1})(u) \in \mathcal{V}$, we have $\det(\psi_v) \neq 0$, which implies that $\det(\psi_v) > 0$. So for such x and v , $\text{ind}_x(s_u) = \text{sgn}(\psi_v) = 1$. By Theorem 3.4, we have

$$\begin{aligned} N(u, \mathcal{V}) &= \#Z(s_u) = \sum_{x \in Z(s_u)} \text{ind}_x(s_u) \\ &= I(s_u, s_0) = e(E_{\mathcal{U}}/E_{\mathcal{V}})([G/T]) = C_{\mathcal{V}}. \end{aligned}$$

□

Proof of Theorem 4.4. “(1) \Rightarrow (2)”. This follows directly from Theorem 4.2.

“(2) \Rightarrow (3)”. Since the map $G \times \mathcal{U} \rightarrow \mathcal{U}$, $(g, u) \mapsto \rho(g)(u)$ is transversal to \mathcal{V} , by the transversality theorem, there exists $u \in \mathcal{U}$ such that $\mathcal{O}_u \pitchfork \mathcal{V}$. Since \mathcal{V} is universal, $\mathcal{O}_u \cap \mathcal{V} \neq \emptyset$. So we may choose $v \in \mathcal{O}_u \cap \mathcal{V}$ and we have $\mathcal{O}_u \pitchfork_v \mathcal{V}$. So $\det(\psi_v) \neq 0$.

“(3) \Rightarrow (1)”. Choose $v_0 \in \mathcal{V}$ such that $\det(\psi_{v_0}) \neq 0$. Then $\mathcal{O}_{v_0} \pitchfork_{v_0} \mathcal{V}$. Consider the map $F : G \times \mathcal{V} \rightarrow \mathcal{U}$, $F(g, v) = \rho(g)(v)$. Then $(dF)_{(e, v_0)}$ is surjective. Hence the image $\text{Im}(F)$ of F contains an open neighborhood of v_0 in \mathcal{U} . By the transversality theorem, we can choose $u \in \text{Im}(F)$ such that $\mathcal{O}_u \pitchfork \mathcal{V}$. Since $u \in \text{Im}(F)$, $\mathcal{O}_u \cap \mathcal{V} \neq \emptyset$. So $Z(s_u)$ is not empty. By Lemma 4.6, $|C_{\mathcal{V}}| = N(u, \mathcal{V}) > 0$. □

Now we give an example for which the assumption of Theorem 4.4 holds. Let $\mathfrak{m}_{\mathbb{C}}$ be the complexification of \mathfrak{m} , and let $\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Phi^{\pm}} \mathfrak{g}_{\alpha}$, where $\Phi^{-} = -\Phi^{+}$. Then $\mathfrak{m}_{\mathbb{C}} = \mathfrak{n}^{+} \oplus \mathfrak{n}^{-}$. Recall that \mathcal{W} can be equipped with the complex structure $J_{\mathcal{W}}$ as in (3.2). Then the map $\psi_v : \mathfrak{m} \rightarrow \mathcal{W}$ extends uniquely to a complex linear map $(\psi_v)_{\mathbb{C}} : \mathfrak{m}_{\mathbb{C}} \rightarrow (\mathcal{W}, J_{\mathcal{W}})$ for all $v \in \mathcal{V}$.

Remark 4.2. As in Remark 3.1, if \mathcal{U} admits a complex structure such that the G -action is complex linear, and if \mathcal{V}, \mathcal{W} are complex subspaces of \mathcal{U} , then $J_{\mathcal{W}}$ can be chosen to be the same as the original complex structure on \mathcal{W} . In this case, the homomorphism $d\rho : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{U})$ extends uniquely to a complex homomorphism $(d\rho)_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{U})$, and we have

$$(\psi_v)_{\mathbb{C}}(X) = -P_{\mathcal{W}}(d\rho)_{\mathbb{C}}(X)(v)$$

for $v \in \mathcal{V}$ and $X \in \mathfrak{m}_{\mathbb{C}}$.

Corollary 4.7. *Let the notation be as above. Suppose that, with respect to the complex structure $J_{\mathcal{W}}$, there is a complex decomposition $\mathcal{W} = \mathcal{W}^{+} \oplus \mathcal{W}^{-}$ such that $(\psi_v)_{\mathbb{C}}(\mathfrak{n}^{\pm}) \subset \mathcal{W}^{\pm}$ for all $v \in \mathcal{V}$. Then \mathcal{V} is universal if and only if $C_{\mathcal{V}} \neq 0$. Furthermore, if $u \in \mathcal{U}$ and $\mathcal{O}_u \pitchfork \mathcal{V}$, then $N(u, \mathcal{V}) = |C_{\mathcal{V}}|$.*

Proof. By Theorem 4.4 and Lemma 4.6, it suffices to show that $\det(\psi_v)$ does not take both positive and negative values on \mathcal{V} .

Let $J'_{\mathcal{W}}$ be the complex structure on \mathcal{W} defined by

$$\begin{cases} J'_{\mathcal{W}}(w) = J_{\mathcal{W}}(w), & w \in \mathcal{W}^{+}; \\ J'_{\mathcal{W}}(w) = -J_{\mathcal{W}}(w), & w \in \mathcal{W}^{-}. \end{cases}$$

Then for $\alpha \in \Phi^+$,

$$\begin{aligned}
\psi_v J_{\mathfrak{m}}(X_\alpha - X_{-\alpha}) &= \psi_v(\sqrt{-1}(X_\alpha + X_{-\alpha})) \\
&= (\psi_v)_{\mathbb{C}}(\sqrt{-1}X_\alpha) + (\psi_v)_{\mathbb{C}}(\sqrt{-1}X_{-\alpha}) \\
&= J_{\mathcal{W}}(\psi_v)_{\mathbb{C}}(X_\alpha) + J_{\mathcal{W}}(\psi_v)_{\mathbb{C}}(X_{-\alpha}) \\
&= J'_{\mathcal{W}}(\psi_v)_{\mathbb{C}}(X_\alpha) - J'_{\mathcal{W}}(\psi_v)_{\mathbb{C}}(X_{-\alpha}) \\
&= J'_{\mathcal{W}}\psi_v(X_\alpha - X_{-\alpha}), \\
\psi_v J_{\mathfrak{m}}(\sqrt{-1}(X_\alpha + X_{-\alpha})) &= -\psi_v(X_\alpha - X_{-\alpha}) \\
&= J'_{\mathcal{W}}J'_{\mathcal{W}}\psi_v(X_\alpha - X_{-\alpha}) \\
&= J'_{\mathcal{W}}\psi_v J_{\mathfrak{m}}(X_\alpha - X_{-\alpha}) \\
&= J'_{\mathcal{W}}\psi_v(\sqrt{-1}(X_\alpha + X_{-\alpha})).
\end{aligned}$$

So the map $\psi_v : \mathfrak{m} \rightarrow \mathcal{W}$ is complex linear with respect to the complex structures $J_{\mathfrak{m}}$ and $J'_{\mathcal{W}}$. Hence with respect to the basis (3.6) of \mathfrak{m} and the basis $\{w'_1, J'_{\mathcal{W}}w'_1, \dots, w'_m, J'_{\mathcal{W}}w'_m\}$ of \mathcal{W} , the matrix of ψ_v has nonnegative determinant, where $\{w'_1, \dots, w'_m\}$ is any complex basis of $(\mathcal{W}, J'_{\mathcal{W}})$. So with respect to the basis (3.6) of \mathfrak{m} and the basis (4.2) of \mathcal{W} , either $\det(\psi_v) \geq 0$ for all $v \in \mathcal{V}$, or $\det(\psi_v) \leq 0$ for all $v \in \mathcal{V}$. \square

5. GENERALIZATIONS OF SCHUR'S TRIANGULARIZATION THEOREM

The classical Schur triangularization theorem asserts that for the complexified adjoint representation of $U(n)$ in $\mathbb{M}(n, \mathbb{C})$, the subspace of $\mathbb{M}(n, \mathbb{C})$ consisting of all upper (or lower) triangular matrices is universal. We generalize this theorem to a whole class of subspaces of $\mathbb{M}(n, \mathbb{C})$ by using the results in the previous sections. We shall also obtain similar generalizations for the conjugation representations of $Sp(n)$ and $SO(n)$ in $\mathbb{M}(n, \mathbb{H})$ and $\mathbb{M}(n, \mathbb{R})$, respectively. To unify the results, we introduce the following definitions.

Definition 5.1. Let n be a positive integer.

(1) An $n \times n$ *zero pattern* is a set

$$I \subset \{(i, j) | i, j \in \{1, \dots, n\}, i \neq j\}$$

of cardinality $n(n-1)/2$.

(2) The *characteristic polynomial* λ_I of an $n \times n$ zero pattern I is

$$\lambda_I(x) = \prod_{(i,j) \in I} (x_i - x_j) \in \mathbb{R}[x_1, \dots, x_n].$$

(3) The *characteristic number* of I is $C_I = \langle \lambda_I, \lambda_0 \rangle$, where λ_0 is the polynomial (2.6), and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}[x_1, \dots, x_n]$ for which the set of monomials $\{x_1^{m_1} \dots x_n^{m_n}\}$ is an orthonormal basis.

It is obvious that λ_0 is the characteristic polynomial of the *upper triangular* zero pattern

$$I_0 = \{(i, j) | 1 \leq i < j \leq n\},$$

and that $C_{I_0} = \langle \lambda_0, \lambda_0 \rangle = n!$.

We first consider the complex case. For $A \in \mathbb{M}(n, \mathbb{C})$, let A_{ij} denote the (i, j) -th entry of A .

Theorem 5.1. *Consider the complexified adjoint representation of $U(n)$ in $\mathbb{M}(n, \mathbb{C})$. Let I be an $n \times n$ zero pattern. If $C_I \neq 0$, then the subspace*

$$\mathbb{M}_I(n, \mathbb{C}) = \{A \in \mathbb{M}(n, \mathbb{C}) \mid A_{ij} = 0 \text{ for } (i, j) \in I\}$$

is universal.

Proof. We keep the notation from Example 2.1. For $(i, j) \in I$, let

$$\mathcal{W}_{ij} = \{A \in \mathbb{M}(n, \mathbb{C}) \mid A_{kl} = 0 \text{ if } (k, l) \neq (i, j)\}.$$

Then \mathcal{W}_{ij} is T -invariant and T acts on \mathcal{W}_{ij} via the character $x_i - x_j$ (identified with $t \mapsto t_i t_j^{-1}$). The subspace $\mathcal{W} = \bigoplus_{(i,j) \in I} \mathcal{W}_{ij}$ of $\mathbb{M}(n, \mathbb{C})$ is complementary to $\mathcal{V} = \mathbb{M}_I(n, \mathbb{C})$. Note that \mathcal{V} has optimal dimension. By Remark 3.1, we have

$$f_{\mathcal{V}}(x) = \prod_{(i,j) \in I} (x_i - x_j) = \lambda_I(x),$$

and

$$C_{\mathcal{V}} = \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} |S_n| = \frac{\langle \lambda_I, \lambda_0 \rangle}{\langle \lambda_0, \lambda_0 \rangle} n! = C_I.$$

So by Theorem 4.2, \mathcal{V} is universal. \square

Note that $\mathbb{M}_{I_0}(n, \mathbb{C})$ consists of all lower triangular matrices. So if $I = I_0$, the above theorem reduces to the classical Schur triangularization theorem.

We say that a zero pattern I is *bitriangular* if $(i_0, j_0) \in I$ and $(j_0 - i_0)(i_0 - j_0 + j - i) > 0$ imply that $(i, j) \in I$. If $i_0 < j_0$ (resp. $i_0 > j_0$), this condition says that $(i, j) \in I$ whenever $j - i > j_0 - i_0$ (resp. $i - j > i_0 - j_0$). Using Corollary 4.7, we can prove the following theorem.

Theorem 5.2. *Let I be a bitriangular zero pattern. Then $\mathbb{M}_I(n, \mathbb{C})$ is universal if and only if $C_I \neq 0$. Furthermore, if the $U(n)$ -orbit of $A \in \mathbb{M}(n, \mathbb{C})$ is transversal to $\mathbb{M}_I(n, \mathbb{C})$, then $N(A, \mathbb{M}_I(n, \mathbb{C})) = |C_I|$.*

Proof. In the notation of Corollary 4.7, \mathfrak{n}^+ (resp. \mathfrak{n}^-) is the set of all strictly upper (resp. lower) triangular matrices in $\mathbb{M}(n, \mathbb{C})$. Let $\mathcal{V} = \mathbb{M}_I(n, \mathbb{C})$, and let \mathcal{W} be the subspace defined in the proof of Theorem 5.1 and $\mathcal{W}^{\pm} = \mathcal{W} \cap \mathfrak{n}^{\pm}$. By Remark 4.2, for $v \in \mathcal{V}$, we have $(\psi_v)_{\mathbb{C}}(X) = P_{\mathcal{W}}([v, X])$, $X \in \mathfrak{m}_{\mathbb{C}}$. Since I is bitriangular, it is easy to see that $P_{\mathcal{W}}([\mathcal{V}, \mathfrak{n}^{\pm}]) \subset \mathcal{W}^{\pm}$. So $(\psi_v)_{\mathbb{C}}(\mathfrak{n}^{\pm}) \subset \mathcal{W}^{\pm}$ for all $v \in \mathcal{V}$, and the theorem follows from Corollary 4.7. \square

Remark 5.1. For the complexified adjoint representation of $U(n)$ in $\mathbb{M}(n, \mathbb{C})$, the number $N(A, \mathbb{M}_I(n, \mathbb{C}))$ of flags sending A into $\mathbb{M}_I(n, \mathbb{C})$ can be interpreted as follows. By viewing A as an operator on \mathbb{C}^n and a flag as an ordered n -tuple (l_1, \dots, l_n) of mutually orthogonal complex lines in \mathbb{C}^n , $N(A, \mathbb{M}_I(n, \mathbb{C}))$ is the number of flags (l_1, \dots, l_n) such that with respect to an ordered basis $\{e_1, \dots, e_n\}$ of \mathbb{C}^n with $e_i \in l_i$, the matrix of A is in $\mathbb{M}_I(n, \mathbb{C})$.

Now we consider the quaternionic case. For a matrix $A \in \mathbb{M}(n, \mathbb{H})$, let A_{ij} denote the (i, j) -th entry of A .

Theorem 5.3. *Consider the conjugation representation of $Sp(n)$ in $\mathbb{M}(n, \mathbb{H})$. Let I be an $n \times n$ zero pattern. If $C_I \neq 0$, then the subspace*

$$\mathbb{M}_I(n, \mathbb{H}) = \{A \in \mathbb{M}(n, \mathbb{H}) \mid A_{ij} = 0 \text{ for } (i, j) \in I\}$$

is universal.

Proof. We keep the notation from Example 2.2. Let \mathcal{V} be the subspace consisting of all matrices $A \in \mathbb{M}_I(n, \mathbb{H})$ such that $A_{ii} \in \mathbb{C}$ for $1 \leq i \leq n$. For $(i, j) \in I$, let

$$\mathcal{W}_{ij} = \{A \in \mathbb{M}(n, \mathbb{H}) \mid A_{kl} = 0 \text{ if } (k, l) \neq (i, j)\},$$

and let

$$\mathcal{W}_i = \{A \in \mathbb{M}(n, \mathbb{H}) \mid A_{ii} \in \mathbb{C} \text{ and } A_{kl} = 0 \text{ for } (k, l) \neq (i, i)\}$$

for $1 \leq i \leq n$. Then the subspace

$$\mathcal{W} = \left(\bigoplus_{(i,j) \in I} \mathcal{W}_{ij} \right) \oplus \left(\bigoplus_{i=1}^n \mathcal{W}_i \right)$$

is complementary to \mathcal{V} in $\mathbb{M}(n, \mathbb{H})$. It is easy to see that each T -invariant subspace \mathcal{W}_{ij} can be decomposed as the direct sum of two T -irreducible subspaces and, with respect to suitable orientations, they are equivalent to the characters $x_i + x_j$ and $x_i - x_j$ of T . Moreover, each \mathcal{W}_i is T -irreducible and, with respect to suitable orientation, it is equivalent to the character $2x_i$ of T . So we have

$$f_{\mathcal{V}}(x) = 2^n \prod_{(i,j) \in I} (x_i^2 - x_j^2) \prod_{i=1}^n x_i = 2^n \lambda_I(x_1^2, \dots, x_n^2) \prod_{i=1}^n x_i.$$

Note that \mathcal{V} has optimal dimension, and the endomorphism of $\mathbb{R}[x_1, \dots, x_n]$ sending $f(x_1, \dots, x_n)$ to $f(x_1^2, \dots, x_n^2) \prod_{i=1}^n x_i$ is isometric. So

$$C_{\mathcal{V}} = \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} |(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n| = \frac{\langle \lambda_I, \lambda_0 \rangle}{\langle \lambda_0, \lambda_0 \rangle} 2^n n! = 2^n C_I.$$

By Theorem 4.2, if $C_I \neq 0$ then \mathcal{V} is universal, and *a fortiori* $\mathbb{M}_I(n, \mathbb{H})$ is universal. \square

For the real case, we first consider the case of $\mathbb{M}(2n, \mathbb{R})$. We partition a matrix $A \in \mathbb{M}(2n, \mathbb{R})$ into 2×2 blocks and denote its (i, j) -th block entry by $a_{ij}(A)$.

Theorem 5.4. *Consider the conjugation representation of $SO(2n)$ in $\mathbb{M}(2n, \mathbb{R})$. Let I be an $n \times n$ zero pattern. If $C_I \neq 0$, then the subspace*

$$\mathbb{M}_I(2n, \mathbb{R}) = \{A \in \mathbb{M}(2n, \mathbb{R}) \mid a_{ij}(A) = 0 \text{ for } (i, j) \in I\}$$

is universal.

Proof. We keep the notation from Example 2.3. For $(i, j) \in I$, let

$$\mathcal{W}_{ij} = \{A \in \mathbb{M}(2n, \mathbb{R}) \mid a_{kl}(A) = 0 \text{ if } (k, l) \neq (i, j)\}.$$

Then $\mathcal{W} = \bigoplus_{(i,j) \in I} \mathcal{W}_{ij}$ is a subspace of $\mathbb{M}(2n, \mathbb{R})$ complementary to $\mathcal{V} = \mathbb{M}_I(2n, \mathbb{R})$. As in the proof of Theorem 5.3, each \mathcal{W}_{ij} decomposes as the direct sum of two T -irreducible subspaces, and with suitable orientations, they are orientation preserving equivalent to the characters $x_i + x_j$ and $x_i - x_j$ of T , respectively. So we have

$$f_{\mathcal{V}}(x) = \prod_{(i,j) \in I} (x_i^2 - x_j^2) = \lambda_I(x_1^2, \dots, x_n^2).$$

Note that \mathcal{V} has optimal dimension, and the endomorphism of $\mathbb{R}[x_1, \dots, x_n]$ sending $f(x_1, \dots, x_n)$ to $f(x_1^2, \dots, x_n^2)$ is isometric. So

$$C_{\mathcal{V}} = \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} |(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n| = \frac{\langle \lambda_I, \lambda_0 \rangle}{\langle \lambda_0, \lambda_0 \rangle} 2^{n-1} n! = 2^{n-1} C_I.$$

By Theorem 4.2, if $C_I \neq 0$ then \mathcal{V}_I is universal. \square

For the case of $\mathbb{M}(2n+1, \mathbb{R})$, we partition a matrix $A \in \mathbb{M}(2n+1, \mathbb{R})$ into blocks $a_{ij}(A)$ of size $(2 - \delta_{i,n+1}) \times (2 - \delta_{j,n+1})$, where $\delta_{i,j}$ is the Kronecker symbol.

Theorem 5.5. *Consider the conjugation representation of $SO(2n+1)$ in $\mathbb{M}(2n+1, \mathbb{R})$. Let J be an $(n+1) \times (n+1)$ zero pattern such that exactly one of $(i, n+1)$ and $(n+1, i)$ is in J for any $i \leq n$, and let $I = \{(i, j) \in J | i, j \leq n\}$. If $C_I \neq 0$, then the subspace*

$$\mathbb{M}_J(2n+1, \mathbb{R}) = \{A \in \mathbb{M}(2n+1, \mathbb{R}) | a_{ij}(A) = 0 \text{ for } (i, j) \in J\}$$

is universal.

Proof. The subspace

$$\mathcal{W} = \{A \in \mathbb{M}(2n+1, \mathbb{R}) | a_{ij}(A) = 0 \text{ for } (i, j) \notin J\}$$

of $\mathbb{M}(2n+1, \mathbb{R})$ is complementary to $\mathcal{V} = \mathbb{M}_J(2n+1, \mathbb{R})$. By choosing a decomposition of \mathcal{W} and suitable orientations on the direct summands, we compute

$$f_{\mathcal{V}}(x) = \prod_{(i,j) \in I} (x_i^2 - x_j^2) \prod_{i=1}^n x_i = \lambda_I(x_1^2, \dots, x_n^2) \prod_{i=1}^n x_i,$$

and

$$C_{\mathcal{V}} = \frac{\langle f_{\mathcal{V}}, f_0 \rangle}{\langle f_0, f_0 \rangle} |(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n| = \frac{\langle \lambda_I, \lambda_0 \rangle}{\langle \lambda_0, \lambda_0 \rangle} 2^n n! = 2^n C_I.$$

So if $C_I \neq 0$, then by Theorem 4.2, \mathcal{V} is universal. \square

Remark 5.2. It is easy to see that the condition of Proposition 4.3 is satisfied for all representations considered in this section.

We say that an $n \times n$ zero pattern I is *simple* if I contains exactly one of (i, j) and (j, i) for any $i, j \in \{1, \dots, n\}$ with $i \neq j$.

Theorem 5.6. *Let I (resp. J) be a simple $n \times n$ (resp. $(n+1) \times (n+1)$) zero pattern. Then in the context of Theorem 5.1 (resp. 5.3, 5.4, 5.5) the subspace $\mathbb{M}_I(n, \mathbb{C})$ (resp. $\mathbb{M}_I(n, \mathbb{H})$, $\mathbb{M}_I(2n, \mathbb{R})$, $\mathbb{M}_J(2n+1, \mathbb{R})$) is universal.*

Proof. We have

$$\lambda_I(x) = \prod_{(i,j) \in I} (x_i - x_j) = \pm \prod_{(i,j) \in I_0} (x_i - x_j) = \pm \lambda_0(x).$$

So $C_I = \langle \lambda_I, \lambda_0 \rangle = \pm \langle \lambda_0, \lambda_0 \rangle = \pm n!$, and the assertion follows from the above theorems. \square

The problem of universality of the subspace in the following example was raised in [7] but seems difficult to prove by other methods.

Example 5.1. Let $n = 3$. Consider the cyclic zero pattern

$$I = \{(1, 3), (2, 1), (3, 2)\}.$$

Since I is a simple zero pattern, $\mathbb{M}_I(3, \mathbb{C})$ is universal with respect to the conjugation representation of $U(3)$ in $\mathbb{M}(3, \mathbb{C})$. \square

Example 5.2 ([23, 5]). Let $n = 4$. Consider the bitriangular zero pattern

$$I = \{(1, 3), (1, 4), (2, 4), (3, 1), (4, 1), (4, 2)\}.$$

Its characteristic polynomial is

$$\lambda_I(x) = -(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_4)^2,$$

and then $C_I = \langle \lambda_I, \lambda_0 \rangle = -12$. So $\mathbb{M}_I(4, \mathbb{C})$ is universal with respect to the conjugation representation of $U(4)$ in $\mathbb{M}(4, \mathbb{C})$. Since I is bitriangular, for $A \in \mathbb{M}(n, \mathbb{C})$ with $\mathcal{O}_A \cap \mathbb{M}_I(4, \mathbb{C})$ we have $N(A, \mathbb{M}_I(4, \mathbb{C})) = 12$. \square

Using the same method, it is easy to generalize Theorems 5.2 and 5.6 to arbitrary complex representations of any connected compact Lie group. We state the result for the complexified adjoint representation below and leave the details to the reader.

Theorem 5.7. *Consider the complexified adjoint representation of a connected compact Lie group G in $\mathfrak{g}_{\mathbb{C}}$. Let Ψ be a subset of the root system Φ such that $|\Psi| = |\Phi^+|$, and let $\mathcal{V} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi \setminus \Psi} \mathfrak{g}_{\alpha}$.*

- (1) *If $\Phi = \Psi \cup (-\Psi)$, then \mathcal{V} is universal;*
- (2) *If $\alpha \in \Psi \cap \Phi^{\pm}$ and $\beta \in \Phi^{\pm}$ (with the same choice of signs) imply that $\alpha + \beta \notin \Phi \setminus \Psi$, then \mathcal{V} is universal if and only if $C_{\mathcal{V}} \neq 0$. Furthermore, if $X \in \mathfrak{g}_{\mathbb{C}}$ and $\mathcal{O}_X \cap \mathcal{V}$, then $N(X, \mathcal{V}) = |C_{\mathcal{V}}|$.*

6. CONCLUDING REMARK

In the previous sections, we assumed that $T \subset G_{\mathcal{V}}$, and then constructed the vector bundles $E_{\mathcal{W}} \cong E_{\mathcal{U}}/E_{\mathcal{V}}$ over G/T , and reduced the question of universality of \mathcal{V} to the existence of zeros of certain sections of $E_{\mathcal{W}}$. A sufficient condition for this is that the Euler class $e(E_{\mathcal{W}})$ is nonzero.

This method remains valid if we replace T by any closed subgroup H of $G_{\mathcal{V}}$. The condition that $T \subset G_{\mathcal{V}}$ can be also dropped. More precisely, for any such H , we let \mathcal{W} be a $G_{\mathcal{V}}$ -invariant subspace of \mathcal{U} complementary to \mathcal{V} , and construct the subbundles $E_{H, \mathcal{V}}$ and $E_{H, \mathcal{W}}$ of the trivial bundle $E_{H, \mathcal{U}} = G/H \times \mathcal{U}$ whose fibers at gH are $\rho(g)(\mathcal{V})$ and $\rho(g)(\mathcal{W})$, respectively. Similarly to Theorem 4.2, if certain characteristic class $o(E_{H, \mathcal{W}})$ (e.g. Euler, top Stiefel-Whitney, or top Chern class) of $E_{H, \mathcal{W}}$ is nonzero, which ensures that every smooth section of $E_{H, \mathcal{W}}$ has a zero, then \mathcal{V} is universal. For example, if H preserves an orientation on \mathcal{W} , then $E_{H, \mathcal{W}}$ is orientable and we can use the Euler class.

If $H_1 \subset H_2$ are two closed subgroups of $G_{\mathcal{V}}$, then there is a natural quotient map $q : G/H_1 \rightarrow G/H_2$, and it is easy to see that $E_{H_1, \mathcal{W}} \cong q^*(E_{H_2, \mathcal{W}})$. Hence we have $o(E_{H_1, \mathcal{W}}) = q^*(o(E_{H_2, \mathcal{W}}))$. If q^* is not injective, it may happen that $o(E_{H_1, \mathcal{W}}) = 0$ but $o(E_{H_2, \mathcal{W}}) \neq 0$. So by using a larger subgroup H of $G_{\mathcal{V}}$, we may get stronger sufficient condition for the universality of \mathcal{V} .

An interesting case occurs when $\dim \mathcal{V} = \dim \mathcal{U} - \dim G/G_{\mathcal{V}}$ and $T \subset G_{\mathcal{V}}$. We further assume that T is the identity component of $G_{\mathcal{V}}$. Then $G_{\mathcal{V}}/T$ is a subgroup of the Weyl group W . To avoid the trouble with non-orientability, we assume that \mathcal{U} admits a complex structure such that the G -action is complex linear, and that \mathcal{V} and \mathcal{W} are complex subspaces. Then we can use the top Chern class (which is equal to the Euler class of the underlying real bundle). We choose $H_1 = T$, $H_2 = G_{\mathcal{V}}$. Then $\dim G/T = \dim G/G_{\mathcal{V}} = 2m$, and $c_{\text{top}}(E_{T, \mathcal{W}}) \in H^{2m}(G/T, \mathbb{Z})$, $c_{\text{top}}(E_{G_{\mathcal{V}}, \mathcal{W}}) \in H^{2m}(G/G_{\mathcal{V}}, \mathbb{Z})$. It is easy to prove that $G/G_{\mathcal{V}}$ is non-orientable if and only if $G_{\mathcal{V}}/T$ contains an odd element of W . From algebraic topology, if $G/G_{\mathcal{V}}$

is orientable, then $H^{2m}(G/G_V, \mathbb{Z}) \cong \mathbb{Z}$ and $q^* : H^{2m}(G/G_V, \mathbb{Z}) \rightarrow H^{2m}(G/T, \mathbb{Z})$ is injective. So we obtain the same information from the top Chern classes by using T and G_V . But if G/G_V is non-orientable, then $H^{2m}(G/G_V, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $q^* : H^{2m}(G/G_V, \mathbb{Z}) \rightarrow H^{2m}(G/T, \mathbb{Z})$ is equal to zero. In this case we always have $c_{\text{top}}(E_{T,W}) = q^*(c_{\text{top}}(E_{G_V,W})) = 0$, but it may happen that $c_{\text{top}}(E_{G_V,W}) \neq 0$, and if this is the case, we get more information than by using T and we can prove the universality of \mathcal{V} . We demonstrate this in the following simple example.

Consider the complexified adjoint representation of $G = SU(2)$ in $\mathcal{U} = \mathfrak{sl}(2, \mathbb{C})$. Let $\mathcal{V} = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$. Then $G_V = N_G(T)$. Let $\mathcal{W} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a \in \mathbb{C} \right\}$. Since $G_V/T = W$ contains an odd element, $e(E_{T,W}) = 0$. Indeed, $E_{T,W}$ is the trivial complex line bundle over $G/T \cong S^2$. But it is easy to see that $E_{G_V,W}$, as a complex line bundle over $G/G_V \cong \mathbb{R}P^2$, is equivalent to the complexification of the tautological real line bundle over $\mathbb{R}P^2$, which has nonzero first Chern class. So $c_{\text{top}}(E_{G_V,W}) \neq 0$ and \mathcal{V} is universal.

The above consideration motivates us to ask the following question.

Question. Suppose that $\dim \mathcal{V} = \dim \mathcal{U} - \dim G/G_V$ and $T \subset G_V$. Does the universality of \mathcal{V} imply that certain obstruction class of $E_{G_V,W}$, for the existence of non-vanishing sections, is nonzero?

We remark that if the condition $\dim \mathcal{V} = \dim \mathcal{U} - \dim G/G_V$ or $T \subset G_V$ is dropped, then in general the answer is negative. This can be seen from the following counter-examples.

For the case of $T \not\subset G_V$, consider the representation $e^{i\theta} \mapsto \text{diag}(e^{i\theta}, e^{2i\theta})$ of $U(1)$ in \mathbb{C}^2 . We view \mathbb{C}^2 as a real vector space, and let $\mathcal{V} = \{(a, b) \in \mathbb{C}^2 \mid \text{Re}(a + b) = 0\}$. Then G_V is trivial, and $E_{G_V,W}$ is a trivial real line bundle over $U(1)$. But it is easy to see that \mathcal{V} is universal.

For the case of $\dim \mathcal{V} > \dim \mathcal{U} - \dim G/G_V$, we can consider the complexified adjoint representation of any semisimple compact group G of rank greater than 1 in $\mathfrak{g}_{\mathbb{C}}$. Let X be a regular element in \mathfrak{t} , let $\mathcal{W} = \mathbb{C}X$, and let $\mathcal{V} = \mathcal{W}^\perp$ with respect to the Killing form of $\mathfrak{g}_{\mathbb{C}}$. Then $G_V = T$, and the above inequality holds. It is easy to see that $E_{G_V,W}$ is a trivial complex line bundle over G/T . But since \mathcal{V} contains the sum of all root spaces in $\mathfrak{g}_{\mathbb{C}}$, \mathcal{V} is universal by [8], Theorem 3.4.

REFERENCES

- [1] I. N. Bernstein, I. M. Gel'fand, S. I. Gel'fand, *Schubert cells and cohomology of the spaces G/P* , Russian Math. Surveys, 28 (1973), no. 3, 1–26.
- [2] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2), 57 (1953), 115–207.
- [3] R. Bott, L. Tu, *Differential forms in algebraic topology*, Springer-Verlag, New York-Berlin, 1982.
- [4] D. Cox, J. Little, D. O'Shea, *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*, Springer, New York, 2007.
- [5] K.R. Davidson, D.Ž. Đoković, *Tridiagonal forms in low dimensions*, Linear Algebra Appl., 407 (2005), 169–188.
- [6] D.Ž. Đoković, *Universal zero patterns for simultaneous similarity of several matrices*, Oper. Matrices, 1 (2007), no. 1, 113–119.
- [7] D.Ž. Đoković, C.R. Johnson, *Unitarily achievable zero patterns and traces of words in A and A^** , Linear Algebra Appl., 421 (2007), no. 1, 63–68.
- [8] D.Ž. Đoković, T.-Y. Tam, *Some questions about semisimple Lie groups originating in matrix theory*, Canad. Math. Bull., 46 (2003), no. 3, 332–343.

- [9] J. Dupont, *Curvature and characteristic classes*, Lecture Notes in Mathematics, Vol. 640, Springer-Verlag, Berlin-New York, 1978.
- [10] A. G. Élashvili, *Canonical form and stationary subalgebras of points of general position for simple linear Lie groups*, Funct. Anal. Appl., 6 (1972), no. 1, 44–53.
- [11] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, 35, Cambridge University Press, Cambridge, 1997.
- [12] J. Galindo, P. de la Harpe, T. Vust, *Two observations on irreducible representations of groups*, J. Lie Theory, 12 (2002), no. 2, 535–538.
- [13] V. Guillemin, A. Pollack, *Differential topology*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.
- [14] S. Helgason, *Groups and geometric analysis*, American Mathematical Society, Providence, RI, 2000.
- [15] J. Holbrook, J.-P. Schoch, *Moving zeros among matrices*, with an appendix by T. Košir and B.A. Sethuraman, Linear Algebra Appl., 424 (2007), no. 1, 83–95.
- [16] W.-C. Hsiang, W.-Y. Hsiang, *Differentiable actions of compact connected classical groups, II*, Ann. of Math. (2), 92 (1970), 189–223.
- [17] R. Kane, *Reflection groups and invariant theory*, Springer-Verlag, New York, 2001.
- [18] K. Kawakubo, *The theory of transformation groups*, Oxford University Press, New York, 1991.
- [19] R. Kiehl, R. Weissauer, *Weil conjectures, perverse sheaves and l -adic Fourier transform*, Springer-Verlag, Berlin, 2001.
- [20] H. Kraft, N.R. Wallach, *On the separation property of orbits in representation spaces*, J. Algebra, 258 (2002), no. 1, 228–254.
- [21] M. Mimura, H. Toda, *Topology of Lie groups, I, II*, American Mathematical Society, Providence, RI, 1991.
- [22] J. Milnor, J. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, 1974.
- [23] V. Pati, *Unitary tridiagonalization in $M(4, \mathbb{C})$* , Proc. Indian Acad. Sci. Math. Sci., 111 (2001), no. 4, 381–397.
- [24] M. Reeder, *On the cohomology of compact Lie groups*, Enseign. Math. (2), 41 (1995), no. 3-4, 181–200.
- [25] I. Schur, *Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen*, Math. Ann., 66 (1909), no. 4, 488–510. See also *Gesammelte Abhandlungen*, Band I, 272–294.
- [26] L. Smith, *Polynomial invariants of finite groups*, A. K. Peters, Ltd., Wellesley, MA, 1995.
- [27] J.-K. Yu, *On the motive and cohomology of a reductive group*, preprint.

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